On the Conversion of Partial Differential Equations

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Z. Naturforsch. **65a,** 896 – 900 (2010); received June 11, 2009 / revised November 20, 2009

This paper outlines the conversion of partial differential equations (PDEs) into the corresponding ordinary differential equations (ODEs) by a complex transformation which is widely used in the exp-function method. The proposed homotopy perturbation method (HPM) is employed to solve the travelling wave solutions. Several examples are given to reveal the reliability and efficiency of the algorithm.

Key words: Homotopy Perturbation Method; Transformation; Partial Differential Equations.

1. Introduction

Recently, He and Wu [1] developed the exp-function method to seek solitary, periodic, and compacton-like solutions of nonlinear differential equations which are widely used in nonlinear sciences [1 – 16]. The exp-function technique [1, 17] uses a transformation $\eta = kx + \omega t$, which converts the given partial differential equations (PDEs) into the corresponding ordinary differential equations (ODEs). The basic motivation of the present study is the elegant coupling of the above transformation and He's homotopy perturbation method (HPM) [2, 6 – 20] for solving PDEs. It is worth mentioning that HPM has been developed by He by merging the standard homotopy and perturbation and is used for finding appropriate solutions of nonlinear problems of physical nature. In our proposed algorithm, the given PDE is converted to the corresponding ODE by using the transformation $\eta = kx + \omega t$. The variational iteration method (VIM) is applied to the re-formulated ODE which gives the solution in terms of transformed variables and the inverse transformation yields the required series of solution. It is observed that the proposed combination is very effective, convenient, and easier to implement; and does not require discretization, linearization, and calculation of the so-called Adomian's polynomials. Numerical results are very encouraging.

2. Homotopy Perturbation Method (HPM) and He's Polynomials

To explain He's homotopy perturbation method, we consider a general equation of the type

$$L(u) = 0, \tag{1}$$

where L is any integral or differential operator. We define a convex homotopy $H(u, p)$ by

$$H(u, p) = (1 - p)F(u) + pL(u), \tag{2}$$

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u, p) = 0, \tag{3}$$

we have

$$H(u, 0) = F(u), \quad H(u, 1) = L(u).$$

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(f, 1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$ continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [2, 6 – 20]. The homotopy perturbation method uses this homotopy parameter p as an expanding parameter to obtain

$$u=\sum_{i=0}^{\infty}p^iu_i=u_0+pu_1+p^2u_2+p^3u_3+\cdots, \quad (4)$$

if $p\to 1$, then (4) corresponds to (2) and becomes the approximate solution of the form

$$f=\lim_{p\to 1}u=\sum_{i=0}^{\infty}u_i. \quad (5)$$

It is well known that series (5) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$; see [1, 6–19]. We assume that (5) has a unique solution. The comparisons of like powers of p give solutions of various orders.

3. Examples

3.1. Example 1

Consider the following Helmholtz equation:

$$\frac{\partial^2u(x,y)}{\partial^2x^2}+\frac{\partial^2u(x,y)}{\partial^2y^2}-u(x,y)=0$$

with initial conditions

$$u(0,y)=y,\quad u_x(0,y)=y+\cosh(y).$$

The exact solution for this problem is given as

$$u(x,y)=y\mathrm{e}^x+y+\cosh(y).$$

Applying the transformation $\eta=x+t$ (by setting $k=\omega=1$), we get the following ODE:

$$2\frac{\mathrm{d}^2u}{\mathrm{d}\eta^2}-u=0$$

with

$$u((\eta)=A,\quad u'(\eta)=B,$$

where A and B are unknown parameters which are subsequently determined by using the initial conditions. Now, we apply the convex homotopy perturbation method

$$u_0+pu_1+p^2u_2+\cdots=$$
$$A+B\eta+p\int_0^{\eta}\int_0^{\eta}\left[\frac{1}{2}\left(u_0+pu_1+p^2u_2+\cdots\right)\right]\mathrm{d}s\,\mathrm{d}s.$$

Comparing the co-efficient of like powers of p yields

$$p^{(0)}:u_0(\eta)=A+B\eta,$$

$$p^{(1)}:u_1(\eta)=A+B\eta+\frac{1}{12}B\eta^3+\frac{1}{4}A\eta^2,$$

$$p^{(2)}:u_2(\eta)=A+B\eta+\frac{1}{480}B\eta^5+\frac{1}{96}A\eta^4+\frac{1}{12}B\eta^3+\frac{1}{4}A\eta^2,$$

$$\vdots$$

The series solution is given by

$$u(\eta)=A+\beta\eta+\frac{1}{480}\beta\eta^5+\frac{1}{96}A\eta^4+\frac{1}{12}\beta\eta^3+\frac{1}{4}A\eta^2+\cdots.$$

The inverse transformation will yield

$$u(x,y)=A+B(x+y)+\frac{1}{480}B(x+y)^5+\frac{1}{96}A(x+y)^4+\frac{1}{12}B(x+y)^3+\frac{1}{4}A(x+y)^2+\cdots$$

and the use of the initial condition gives

$$A=-2\big(2y^3\mathrm{e}^y+y^2+y^2(\mathrm{e}^y)^2-6y^2\mathrm{e}^y+24y\mathrm{e}^y-24\mathrm{e}^y+12+12(\mathrm{e}^y)^2\big)y[\mathrm{e}^y(48+y^4)]^{-1},$$

$$B=6\big[-4y^2\mathrm{e}^y+y^2+y^2(\mathrm{e}^y)^2+2y^3\mathrm{e}^y+8y\mathrm{e}^y+4+4(\mathrm{e}^y)^2\big]\big[\mathrm{e}^y(48+y^4)\big]^{-1}.$$

The solution after two iterations is given by

$$u(x,y)=\big[96y\mathrm{e}^y+8y\mathrm{e}^yx^3+48x-4y^2\mathrm{e}^yx^3-6y^3\mathrm{e}^yx^2+96y\mathrm{e}^yx+2y^3\mathrm{e}^yx^3+4y^4\mathrm{e}^yx^2+2y^5\mathrm{e}^yx+y^2\mathrm{e}^{2y}x^3+2y^3\mathrm{e}^{2y}x^2+y^4\mathrm{e}^{2y}x+24y\mathrm{e}^yx^2+48\mathrm{e}^{2y}x+2y^5\mathrm{e}^y+4\mathrm{e}^{2y}x^3+y^2x^3+2y^3x^2+y^4x+x^3\big]\big[2\mathrm{e}^y(48+y^4)\big]^{-1}.$$

3.2. Example 2

Consider the following Helmholtz equation:

$$\frac{\partial^2u(x,y)}{\partial^2x^2}+\frac{\partial^2u(x,y)}{\partial^2y^2}+8u(x,y)=0$$

with the initial conditions

$$u(0,y)=\sin(2y),\quad u_x(0,y)=0.$$

The exact solution for this problem is

$$u(x,y)=\cos(2x)\sin(2y).$$

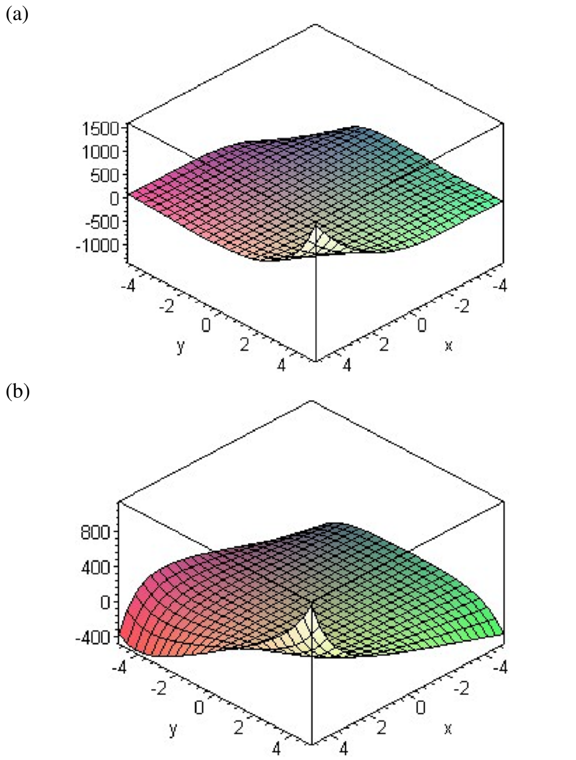

Fig. 1 (colour online). Solution by the proposed algorithm (a) and exact solution (b) of Example 1.

Applying the transformation $\eta = x+t$ (by setting $k = \omega = 1$), we get

$$\frac{\mathrm{d}^2u}{\mathrm{d}\eta^2}+4u=0$$

with

$$u(\eta)=A, \quad u'(\eta)=B.$$

Applying the convex homotopy perturbation method, we get

$$u_0+pu_1+p^2u_2+\cdots = A+B\eta-4p\int_0^\eta\int_0^\eta(u_0+pu_1+p^2u_2+\cdots)\mathrm{d}s\,\mathrm{d}s.$$

Comparing the co-efficient of like powers of p yields

$$p^{(0)}: u_0(\eta)=A+B\eta,$$
$$p^{(1)}: u_1(\eta)=A+B\eta-\frac{2}{3}B\eta^3-2A\eta^2,$$
$$p^{(2)}: u_2(\eta)=A+B\eta+\frac{2}{15}B\eta^5+\frac{2}{3}A\eta^4 -\frac{2}{3}B\eta^3-2A\eta^2,$$
$$\vdots$$

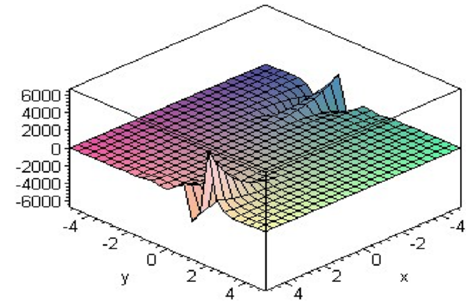

Fig. 2 (colour online). Solution of Example 2.

The series solution is given by

$$u(\eta)=A+B\eta+\frac{2}{15}B\eta^5+\frac{2}{3}A\eta^4-\frac{2}{3}B\eta^3-2A\eta^2+\cdots,$$

the inverse transformation will yield

$$u(x,y)=A+B(x+y)+\frac{2}{15}B(x+y)^5+\frac{2}{3}A(x+y)^4 -\frac{2}{3}B(x+y)^3-2A(x+y)^2+\cdots,$$

and the use of the initial condition gives

$$A=15\frac{\sin(2y)(2y^4-6y^2+3)}{-16y^6+4y^8+45},$$
$$B=-60\frac{y\sin(2y)(-3+2y^2)}{-16y^6+4y^8+45}.$$

The solution after two iterations is given by

$$u(x,y)=\sin(2y)\big[-45-30x^4 2y^4-24xy^5-60y^2x^4 -80y^3x^3-60y^4x^2+16y^3x^5+60y^4x^4+80y^5x^3 +40y^6x^2+90x^2+16y^6-4y^8\big]\big[-16y^6+4y^8+45\big]^{-1}.$$

3.3. Example 3

Consider the homogeneous telegraph equation

$$\frac{\partial^2u(x,t)}{\partial^2x^2}=\frac{\partial^2u(x,t)}{\partial^2t^2}+\frac{\partial u(x,t)}{\partial t}-u(x,y)$$

with initial and boundary conditions

$$I.C.: \quad u(x,0)=\mathrm{e}^x,\ u_x(x,0)=-2\mathrm{e}^x.$$
$$B.C.: \quad u(0,t)=\mathrm{e}^{-2t},\ u_x(0,t)=\mathrm{e}^{-2t},$$

The exact solution for this problem is

$$u(x,t)=\mathrm{e}^{x-2t},$$

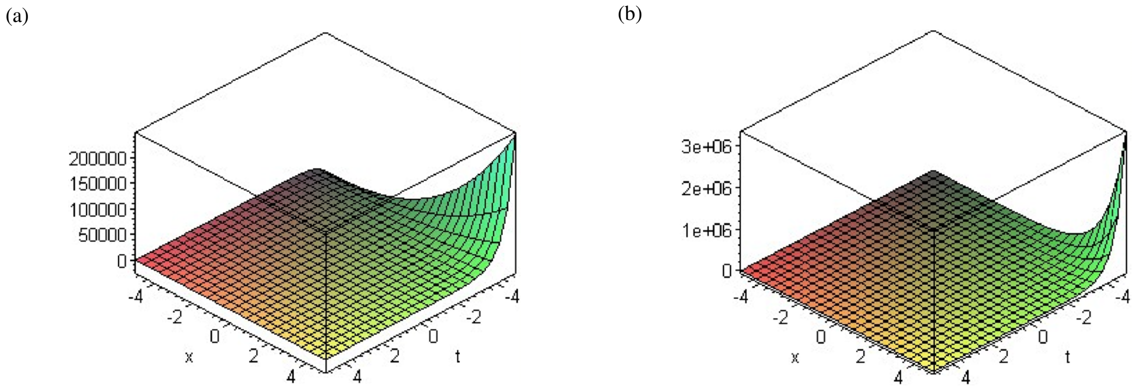

Fig. 3 (colour online). Solution by the proposed algorithm (a) and exact solution (b) of Example 3.

where A and B are unknown parameters which are subsequently determined by using the initial conditions. Applying the convex homotopy perturbation method we get

$$\begin{aligned} & u_0+pu_1+p^2u_2+\cdots \\ & = A+B\eta+p\int_0^\eta\int_0^\eta\left[\left(\frac{\partial^2 u_0}{\partial s^2}+p\frac{\partial^2 u_1}{\partial s^2}+\cdots\right)\right. \\ & \quad \left.+\left(\frac{\partial u_0}{\partial s}+p\frac{\partial u_1}{\partial s}+\cdots\right)-(u_0+pu_1+\cdots)\right]\mathrm{d}s\,\mathrm{d}s. \end{aligned}$$

Comparing the co-efficient of like powers of p, following approximants are obtained:

$$\begin{aligned} & p^{(0)}: u_0(\eta)=A+B\eta, \\ & p^{(1)}: u_1(\eta)=A+B\eta-\frac{1}{18}B\eta^3+\frac{1}{6}A\eta^2+\frac{1}{3}B\eta^2, \\ & p^{(2)}: u_2(\eta)=A+B\eta+\frac{2}{1080}B\eta^5+\frac{1}{216}A\eta^4+\frac{1}{54}B\eta^4 \\ & \qquad +\frac{7}{54}B\eta^3+\frac{1}{27}A\eta^3+\frac{1}{6}A\eta^2+\frac{1}{3}B\eta^2, \\ & \vdots \end{aligned}$$

The series solution is given by

$$\begin{aligned} u(\eta) = {} & A+B\eta+\frac{2}{1080}B\eta^5+\frac{1}{216}A\eta^4+\frac{1}{54}B\eta^4 \\ & +\frac{7}{54}B\eta^3+\frac{1}{27}A\eta^3+\frac{1}{6}A\eta^2+\frac{1}{3}B\eta^2+\cdots, \end{aligned}$$

the inverse transformation would yield

$$\begin{aligned} u(x,t) = {} & A+B(x-2t)+\frac{2}{1080}B(x-2t)^5 \\ & +\frac{1}{216}A(x-2t)^4+\frac{1}{54}B(x-2t)^4 \\ & +\frac{7}{54}B(x-2t)^3+\frac{1}{27}A(x-2t)^3 \\ & +\frac{1}{6}A(x-2t)^2+\frac{1}{3}B(x-2t)^2+\cdots, \end{aligned}$$

and the use of the initial condition gives

$$\begin{aligned} A &= 3\frac{\mathrm{e}^{-2t}(9+6t-6t^2+4t^3)}{4t^4+27-36t}, \\ B &= 9\frac{\mathrm{e}^{-2t}(3+2t+2t^2)}{4t^4+27-36t}. \end{aligned}$$

The solution after two iterations is given by

$$\begin{aligned} u(x,t) = {} & \mathrm{e}^{-2t}\left[54+3x^3-72tx+27x^2+2tx^3-6t^2x^2\right. \\ & \left.+2t^2x^3-8t^3x^2+8t^4x+54x-72t+8t^4\right] \\ & \cdot\left[2(4t^4+27-36t)\right]^{-1}. \end{aligned}$$

4. Conclusion

In this paper, we applied a reliable combination of HPM and the transformation introduced by He and Wu [1] for solving partial differential equations. It may be concluded that the proposed coupling is very powerful and efficient in finding the analytical solutions for a wide class of PDEs.

Acknowledgement

The author is highly grateful to the referee for his/her very constructive comments.

[1] J. H. He and X. H. Wu, Chaos, Solitons, and Fractals **30**, 700 (2006).
[2] A. Aslanov, Z. Naturforsch. **64a**, 149 (2009).
[3] M. A. Abdou and A. A. Soliman, Phys. D **211**, 1 (2005).
[4] J. Biazar and H. Ghazvini, Comput. Math. Appl. **54**, 1047 (2007).
[5] D. D. Ganji, H. Tari, and M. B. Jooybari, Comput. Math. Appl. **54**, 1018 (2007).
[6] J. H. He, Top. Method Nonlinear Anal. **31**, 205 (2008).
[7] J. H. He, Int. J. Mod. Phys. **10**, 1144 (2006).
[8] J. H. He, Appl. Math. Comput. **156**, 527 (2004).
[9] J. H. He, Int. J. Nonlinear Sci. Numer. Simul. **6**, 207 (2005).
[10] J. H. He, Appl. Math. Comput. **151**, 287 (2004).
[11] J. H. He, Int. J. Nonlinear Mech. **35**, 115 (2000).
[12] J. H. He, Int. J. Mod. Phys. B **22**, 3487 (2008).
[13] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Math. Prob. Eng. (2009), Article ID 234849, DOI: 10.1155/2009/234849.
[14] S. T. Mohyud-Din and M. A. Noor, Math. Prob. Eng. (2007), Article ID 98602, DOI: 10.1155/2007/98602.
[15] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Int. J. Nonlinear. Sci. Numer. Simul. **10**, 223 (2009).
[16] S. T. Mohyud-Din and M. A. Noor, Z. Naturforsch. **64a**, 157 (2009).
[17] S. D. Zhu, Inter. J. Nonlinear Sci. Numer. Simul. **8**, 461 (2007).
[18] M. A. Noor and S. T. Mohyud-Din, Comput. Math. Appl. **54**, 1101 (2007).
[19] A. Yildirim, Z. Naturforsch. **63a**, 621 (2008).
[20] E. M. E. Zayed, T. A. Nofal, and K. A. Gepreel, Z. Naturforsch. **63a**, 627 (2008).